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ON THE ELLIPTIC HARMONIC BALANCE METHOD FOR MIXED PARITY NON-LINEAR OSCILLATORS

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Mixed parity non-linear oscillators often occur in mechanics. Examples are asymmetric vibrations of pendulums, shells and curved bars [1].

In a paper by Gottlieb [2], a mixed parity autonomous oscillator was investigated using the standard harmonic balance method. His work was motivated by an example conducted by Nayfeh and Mook [3] in which they have cautioned against use of the method when a mixed parity is present in the system, pointing out that for more accuracy, a second harmonic term (as well as a constant) must be taken into account in the solution expression.

In contrast, Gottlieb [2] showed that a second harmonic term is not needed, for large amplitude, to obtain accurate expressions of the frequency. In supporting his affirmation a comparison of the period of oscillations, in a special example, was reported.

In this letter, we apply the harmonic balance method involving the Jacobian elliptic functions [4] to show how the approximation of the period obtained in reference [2], using the trigonometric harmonic balance method, can be significantly improved.

Consider a strongly, mixed parity non-linear oscillator described by the equation

$$\ddot{x} + ax + bx^2 + c^2x^3 + d = 0. \tag{1}$$

By introducing the dependent variable change $x = X - b/(3c^2)$, the quadratic non-linearity in equation (1) is eliminated to obtain

$$\ddot{X} + \left(a - \frac{b^2}{3c^2}\right)X + c^2X^3 = \varepsilon Z, \tag{2}$$

where

$$\varepsilon Z = \frac{ab}{3c^2} - \frac{2b^3}{27c^4} - d. \tag{3}$$

In the case of large amplitude of X, it is assumed that the constant term εZ on the right-hand side of equation (2) is small. In this context, the periodic solutions of equation (2) are approximated here using elliptic harmonic balance [4]. Recently, this elliptic method was successfully used [5, 6] to derive a criterion for homoclinic bifurcation of an autonomous planar system aiming at the collision of the approximate periodic orbit with the saddle instead of considering, as usual, the distance between the separatrix.

The choice of the convenient elliptic functions depends, in fact, on the sign of $(a - b^2/3c^2)$ and on the total energy E.

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We assume a solution of equation (2) in the form

$$X(t) = A\, pq(\omega t; k^2), \tag{4}$$

where A and ω are constants to be determined and pq denotes a convenient Jacobian elliptic function. Substituting equation (4) into equation (2) gives

$$F_1(A, \omega, k^2, \varepsilon, \alpha)\cos\phi + F_2(a, \omega, k^2, \varepsilon, \alpha)\sin\phi + (\text{higher harmonics}) = 0,$$

where ϕ is the generalized circular function of the case being considered and α collectively denotes any parameter which appears in the non-linear function. To determine the unknowns quantities, we then set $F_1 = 0$ and $F_2 = 0$. Hence, for large oscillations about $X = 0$, i.e., for amplitude A sufficiently large that $x = -(b/3c^2) \pm A$ lies on the outside parts of the potential curve

$$V(x) = \frac{c^2x^4}{4} + \frac{bx^3}{3} + \frac{ax^2}{2} + dx. \tag{5}$$

Up to the first order, the elliptic harmonic balance method [4] leads to

$$x(t) = -\frac{b}{3c^2} + A\,\mathrm{cn}(\omega t, k), \tag{6}$$

$$\omega^2 = a - \frac{b^2}{3c^2} + c^2A^2, \qquad k^2 = \frac{c^2A^2}{2\omega^2}. \tag{7, 8}$$

As an example, we consider the same non-linear oscillator equation as in reference [2]:

$$\ddot{x} = x^2 - x^3. \tag{9}$$

As mentioned before, the quadratic term in equation (9) may be eliminated by the dependent variable shift $X = x - (\frac{1}{3})$ to get

$$\ddot{X} - \tfrac{1}{3}X + X^3 = \tfrac{2}{27}. \tag{10}$$

In reference [2] it was supposed that when the amplitude is large, the term on the right-hand side in equation (10) (which now contains the only even parity term) may be neglected. The standard harmonic balance method was used for large (symmetric) oscillations about $X = 0$, to obtain

$$x(t) = \tfrac{1}{3} + A\cos(\omega t), \qquad \omega^2 = \tfrac{3}{4}A^2 - \tfrac{1}{3} \tag{11, 12}$$

satisfying

$$x(0) = \tfrac{1}{3} + A, \qquad \dot{x}(0) = 0, \tag{13}$$

where A is large.

Now using the elliptic harmonic balance method to equation (10), we find for large amplitude, i.e., $A(\omega) > \sqrt{\frac{2}{3}}$, the following approximate solution:

$$x(t) = \tfrac{1}{3} + A\,\mathrm{cn}(\omega t, k), \tag{14}$$

$$\omega^2 = -\tfrac{1}{3} + A^2, \qquad k^2 = \frac{A^2}{2\omega^2}. \tag{15}$$

Comparisons with numerical integration of equation (9) with initial conditions (13) shows that formula (15) leads to excellent approximations for the oscillation period. Indeed, for $A > 3$ the error is about 0·01% using the elliptic harmonic balance and about 2% using the standard harmonic balance. For instance, for $A = 10$, T(equation (12)) = 0·7271, T(equation (15)) = 0·7434 and T(equation (9)) = 0·7435.

The use of elliptic functions provides good approximations of periodic solution and its period, specially for large amplitudes. This clearly shows the superiority of the elliptic balancing to the approaches using circular functions.

REFERENCES

1. A. H. Nayfeh and B. Balachandran 1989 *Applied Mechanics Reviews* **42**, part 2, 175–201. Modal interactions in dynamical and structural systems.
2. H. P. W. Gottlieb 1992 *Journal of Sound and Vibration* **152**, 189–191. On the harmonic balance method for mixed-parity nonlinear oscillators.
3. A. H. Nayfeh and D. T. Mook 1979 *Nonlinear Oscillations*. New York: John Wiley & Sons.
4. J. G. Margalo and J. D. Bejarano 1990 *Journal of Sound and Vibration* **136**, 453–466. Generalized Fourier series and limit cycles of generalized Van der Pol oscillators.
5. M. Belhaq, B. Fiedler and F. Lakrad (in press) *Nonlinear Dynamics*. Homoclinic connections in strongly self-excited nonlinear oscillators: the Melnikov function and the elliptic Lindstedt–Poincaré method.
6. M. Belhaq and F Lakrad (in press) *Chaos Solitons and Fractals*. Prediction of homoclinic bifurcation: the elliptic averaging method.